Conformal Invariance and Cosmic Background Radiation

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The spectrum and statistics of the cosmic microwave background radiation (CMBR) are investigated under the hypothesis that scale invariance of the primordial density fluctuations should be promoted to full conformal invariance. As in the theory of critical phenomena, this hypothesis leads in general to deviations from naive scaling. The spectral index of the two-point function of density fluctuations is given in terms of the quantum trace anomaly and is greater than one, leading to less power at large distance scales than a strict Harrison-Zel'dovich spectrum. Conformal invariance also implies non-gaussian statistics for the higher point correlations and in particular, it completely determines the large angular dependence of the three-point correlations of the CMBR.

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With the discovery of cosmic microwave background anisotropy [1], cosmology has accelerated its transition from a field based largely on speculation to one in which observational data can be brought to bear on our understanding of the universe. As a newly emerging physical science, it is appropriate and may prove fruitful to examine the consequences for cosmology of progress in its more developed sister sciences.

Scale invariance was first introduced into physics in early attempts to understand the apparently universal behavior observed in turbulence and second order phase transitions, which are independent of the particular dynamical details of the system. The gradual refinement and development of this simple idea of universality has led to the theory of critical phenomena, which is remarkable both for its broad applicability and quantitative predictive power [2]. One of the hallmarks of the modern theory of critical phenomena is well-defined logarithmic deviations from naive scaling relations based on engineering dimensions. A second general feature of the theory is the specification of higher point correlation functions of fluctuations according to the requirements of conformal invariance at the critical point [3].

In cosmology scale invariance was first raised to a level of prominence by the pioneering work of Harrison and Zel'dovich on the spectrum of primordial density fluctuations required to produce the observed large scale structure in the universe [4]. In inflationary scenarios density fluctuations with a spectral index n very close to one can be generated from quantum fluctuations at a very early stage in the history of the universe at the threshold of its classical evolution [5]. The most direct probe of these primordial density fluctuations is the cosmic microwave background radiation (CMBR). With the observational data of the CMBR anisotropy providing confirmation of this speculative foundation, the time now seems ripe to go one step further in developing the theoretical framework and apply the considerable lessons of universality in critical phenomena to the universe itself.

In the language of critical phenomena, the observation of Harrison and Zel'dovich that the primordial density fluctuations should be characterized by a spectral index $n = 1$ is equivalent to the statement that the observable giving rise to these fluctuations has engineering or naive scaling dimension $\Delta_0 = 2$. Indeed, because the density fluctuations are related to metric fluctuations by Einstein's equations, this naive scaling dimension simply reflects the fact that the relevant coordinate invariant measure of metric fluctuations is the scalar curvature $\delta R \sim G\delta\rho$, which is second order in derivatives of the metric. Hence, the fluctuations in the density perturbations are tied to the scalar curvature and the two-point correlations of both should behave like $|x-y|^{-4}$, or $|k|^1$ in Fourier space, according to simple dimensional analysis.

One of the principal lessons of the modern understanding of critical phenomena is that naive dimensional analysis does not fix the transformation properties of observables under conformal transformations at the fixed point, but that instead one must expect to find well-defined logarithmic deviations from naive scaling, corresponding to a (generally non-integer) dimension $\Delta \neq \Delta_0$. The deviation from naive scaling $\Delta - \Delta_0$ is the "anomalous" dimension of the observable due to critical fluctuations which may be quantum or statistical in origin. The requirement of conformal invariance then determines the form of the two- and three-point correlation functions of the observable in terms of its dimension Δ, up to an arbitrary amplitude.

Two-point correlations. In the case of the two-point function of two observables $\mathcal{O}_\Delta$ with dimension Δ, conformal invariance requires

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle \sim |x_1 - x_2|^{-2\Delta} \tag{1}$$

at equal times in three dimensional flat spatial coordi-

nates. In Fourier space this becomes

$$G_2(k) \equiv \langle \tilde{\mathcal{O}}_\Delta(k) \tilde{\mathcal{O}}_\Delta(-k) \rangle \sim |k|^{2\Delta-3} . \tag{2}$$

Thus, we define the spectral index of this observable by

$$n \equiv 2\Delta - 3 . \tag{3}$$

In the case that the observable is the primordial density fluctuation $\delta\rho$, and in the classical limit where its anomalous dimension vanishes, $\Delta \to \Delta_0 = 2$, we recover the Harrison-Zel'dovich spectral index of $n = 1$.

In order to convert the power spectrum of primordial density fluctuations to the spectrum of fluctuations in the CMBR at large angular separations we follow the standard treatment [6,7], relating the temperature deviation to the Newtonian gravitational potential φ at the last scattering surface, $\frac{\delta T}{T} \sim \delta\varphi$, which is related to the density perturbation by

$$\nabla^2 \delta\varphi = 4\pi G\, \delta\rho . \tag{4}$$

Hence, in Fourier space

$$\frac{\delta T}{T} \sim \frac{1}{k^2} \frac{\delta\rho}{\rho} , \tag{5}$$

and the two-point function of CMBR temperature fluctuations is determined by the conformal dimension Δ to be

$$C_2(\theta) \equiv \left\langle \frac{\delta T}{T}(\hat{r}_1) \frac{\delta T}{T}(\hat{r}_2) \right\rangle \sim \int d^3k \left(\frac{1}{k^2}\right)^2 G_2(k) e^{ik\cdot r_{12}} \sim \Gamma(2-\Delta)(r_{12}^2)^{2-\Delta} , \tag{6}$$

where $r_{12} \equiv (\hat{r}_1 - \hat{r}_2) r$ is the vector difference between the two positions from which the CMBR photons originate. They are at equal distance r from the observer by the assumption that the photons were emitted at the last scattering surface at equal cosmic time. Since $r_{12}^2 = 2(1-\cos\theta) r^2$, we find then

$$C_2(\theta) \sim \Gamma(2-\Delta)(1-\cos\theta)^{2-\Delta} \tag{7}$$

for arbitrary scaling dimension Δ.

The meaning of the pole in the Γ function at Δ equal to 2 is best understood by expanding the function $C_2(\theta)$ in multipole moments,

$$C_2(\theta) = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell+1) c_\ell^{(2)}(\Delta) P_\ell(\cos\theta) , \tag{8}$$

with

$$c_\ell^{(2)}(\Delta) \sim \Gamma(2-\Delta) \sin\left[\pi(2-\Delta)\right] \frac{\Gamma(\ell+\Delta-2)}{\Gamma(\ell+4-\Delta)} , \tag{9}$$

which shows that the pole singularity appears only in the $\ell = 0$ monopole moment. This singularity occurs due to the integration over the whole space and is just the reflection of the fact that the Laplacian in (4) cannot be inverted on constant functions. Since the CMBR anisotropy is defined by removing the isotropic monopole moment (as well as the dipole moment, which receives a substantial contribution from the proper motion of the earth with respect to the CMBR [7]), the $\ell = 0$ moment does not appear in the sum, and the higher moments of the anisotropic two-point correlation function are well-defined for Δ near 2. Normalizing to the quadrupole moment $c_2^{(2)}(\Delta)$, we find

$$c_\ell^{(2)}(\Delta) = c_2^{(2)}(\Delta) \frac{\Gamma(6-\Delta)}{\Gamma(\Delta)} \frac{\Gamma(\ell+\Delta-2)}{\Gamma(\ell+4-\Delta)} , \tag{10}$$

which is a standard result [7,8]. Indeed, if Δ is replaced by $\Delta_0 = 2$ we obtain $\ell(\ell+1)c_\ell^{(2)}(\Delta_0) = 6c_2^{(2)}(\Delta_0)$, which is the well-known predicted behavior of the lower moments ($\ell \leq 30$) of the CMBR anisotropy where the Sachs-Wolfe effect should dominate.

Up to this point our considerations have been quite general. In order to say something more definite about the expected violations of classical scale invariance we need a specific proposal for their physical origin. This is provided by consideration of the zero-point quantum fluctuations of massless fields, which give rise to the conformal trace anomaly $T^\mu{}_\mu \neq 0$. Such a non-zero trace associated with long-range fields couples to the spin-0 or conformal part of the metric, and cause it to fluctuate as well. These fluctuations of the metric grow logarithmically at distance scales of the order of the horizon and lead to a renormalization group flow to an infrared stable conformal invariant fixed point of gravity [9]. At this fixed point conformal invariance is restored but there are well-defined deviations of the scaling dimensions of observables from their classical values, calculable in terms of the coefficient of the original trace anomaly.

This analysis determines the general form of the conformal dimension of an observable with naive dimension Δ_0 to be [10]

$$\Delta = 4 \frac{\sqrt{1 - 2(4-\Delta_0)/Q^2} - \sqrt{1 - 8/Q^2}}{1 - \sqrt{1 - 8/Q^2}} , \tag{11}$$

where Q^2 is the coefficient of a certain term (the Gauss-Bonnet term) in the trace anomaly, given by Eqn. (13) below. In the limit $Q^2 \to \infty$, the effects of fluctuations in the metric due to the trace anomaly are suppressed and one recovers the classical scaling dimension Δ_0,

$$\Delta = \Delta_0 + \frac{1}{2Q^2} \Delta_0 (4 - \Delta_0) + \cdots \tag{12}$$

Hence, consideration of conformal fluctuations of the metric generated by the trace anomaly of massless fields leads necessarily to well-defined quantum corrections to the naive scaling dimensions of observables in cosmology.

Moreover, in the analysis of physical observables in the conformal sector of gravity, the operator with the lowest non-trivial scaling dimension corresponds, in the semi-classical limit, to the scalar curvature with $\Delta_0 = 2$ [10]. Since the fluctuations which dominate at large distances correspond to observables with lowest scaling dimensions, the conformal factor theory in this limit selects precisely Harrison's original choice.

With $\Delta_0 = 2$, we find a definite prediction for deviations from a strict Harrison-Zel'dovich spectrum according to Eqns. (3) and (11) in terms of the parameter Q^2. The resulting spectral index n is plotted as a function of Q^2 in Fig. 1. It is always greater than 1 (if $8 \leq Q^2 < \infty$), and for large Q^2 it behaves as $n = 1 + \frac{4}{Q^2} + \cdots$. Comparing to the results of the four year COBE DMR data analysis of the power spectrum, $0.9 \lesssim n_{obs} \lesssim 1.5$ [11], we find that $Q^2_{obs} \gtrsim 12.4$ from Fig. 1.

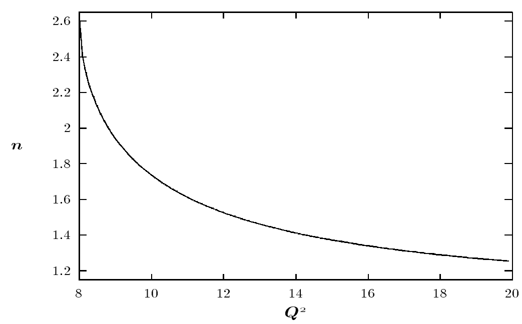

FIG. 1. The spectral index n as a function of Q^2.

From the theoretical side, the value of Q^2 for free conformally invariant fields is known to be [12,13]

$$Q^2 = \frac{1}{180}(N_S + \tfrac{11}{2}N_F + 62N_V - 28) + Q^2_{grav} \ , \quad (13)$$

where N_S, N_F, N_V are the number of free scalars, Weyl fermions and vector fields and Q^2_{grav} is the contribution of spin-2 gravitons, which has not yet been determined unambiguously. The -28 contribution is that of the conformal or spin-0 part of the metric itself. The main theoretical difficulty in determining Q^2_{grav} is that the Einstein theory is neither conformally invariant nor free, so that a method for evaluating the strong infrared effects of spin-2 gravitons must be found that is insensitive to ultraviolet physics. Such an analysis may be possible by numerical methods on the lattice, which would also provide a nontrivial consistency check of the existence of the fixed point with the predicted scaling relations [10]. A purely perturbative computation gives $Q^2_{grav} \simeq 7.9$ for the graviton contribution [13]. Taking this estimate at face value and including all known fields of the Standard Model of particle physics (for which $N_F = 45$ and $N_V = 12$) we find

$$Q^2_{SM} \simeq 13.2 \qquad \text{and} \qquad n \simeq 1.45 \ , \quad (14)$$

which is intriguingly close to the observational bound.

A deviation from the Harrison-Zel'dovich spectrum has implications for galaxy formation as well. Indeed, a determination of Q^2 close to its present observational bound together with the COBE quadrupole normalization of the spectrum at large scales implies more power at shorter subhorizon scales where galaxies formed. For example, for the value of the spectral index $n \simeq 1.45$, the power spectrum has an enhancement factor of $(H_0 \times 20 \text{ Mpc}/2h)^{-0.45} \simeq 4.6$ at the $20h^{-1}$ Mpc distance scale, relative to the $n = 1$ spectrum. This would lead to earlier formation of structure at the galactic and galactic cluster scales than in the case of a primordial $n = 1$ spectrum. However, the form and normalization of the evolved cluster mass function at these scales is very much model dependent and would need to be reanalyzed *ab initio* in each dark matter model to decide if increased power in the primordial spectrum of adiabatic density fluctuations can be reconciled with the observations of the matter anisotropy on this scale [7]. It is noteworthy that the conformal invariant fixed point for gravity predicts a spectral index $n > 1$, while most suggestions for modifying the Harrison-Zel'dovich spectrum such as extended or power law inflation generally lead to $n \leq 1$ [14].

Higher point correlations. Turning now from the two-point function of CMBR fluctuations to higher point correlators, we find a second characteristic and unambiguous prediction of conformal invariance, namely non-gaussian statistics for the CMBR. The first correlator sensitive to this departure from gaussian statistics is the three-point function of the observable $\mathcal{O}_\Delta$, which takes the form [3]

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\rangle \sim \\ |x_1 - x_2|^{-\Delta}|x_2 - x_3|^{-\Delta}|x_3 - x_1|^{-\Delta} \ , \quad (15)$$

or in Fourier space,

$$G_3(k_1, k_2) \sim \int d^3p \ |p|^{\Delta-3} \, |p + k_1|^{\Delta-3} \, |p - k_2|^{\Delta-3} \ \sim \\ \Gamma\left(3(1 - \tfrac{\Delta}{2})\right) \int_0^1 du \int_0^1 dv \ \times \\ \frac{[u(1-u)v]^{1-\frac{\Delta}{2}} \, (1-v)^{-1+\frac{\Delta}{2}}}{\left[u(1-u)k_1^2 + v(1-u)k_2^2 + uv(k_1+k_2)^2\right]^{3(1-\frac{\Delta}{2})}} \ . \quad (16)$$

This three-point function of primordial density fluctuations gives rise to three-point correlations in the CMBR by reasoning precisely analogous as that leading from Eqns. (2) to (6). That is,

$$C_3(\theta_{12}, \theta_{23}, \theta_{31}) \equiv \left\langle \frac{\delta T}{T}(\hat{r}_1)\frac{\delta T}{T}(\hat{r}_2)\frac{\delta T}{T}(\hat{r}_3)\right\rangle \sim \\ \int \frac{d^3k_1 \, d^3k_2}{k_1^2 k_2^2 (k_1 + k_2)^2} G_3(k_1, k_2) e^{ik_1 \cdot r_{13}} e^{ik_2 \cdot r_{23}} \ , \quad (17)$$

where $r_{ij} \equiv (\hat{r}_i - \hat{r}_j)r$ and $r^2_{ij} = 2(1 - \cos\theta_{ij})r^2$.

From the above expressions, it is easy to extract the global scaling of the three-point function in the infrared:

$$G_3(\lambda k_1, \lambda k_2) \sim \lambda^{3(\Delta-2)} G_3(k_1, k_2) \ ,$$
$$C_3 \sim r^{3(2-\Delta)} \ . \tag{18}$$

In the general case of three different angles, the expression for the three-point correlation function (17) is quite complicated, though it can be rewritten in parametric form analogous to (16) to facilitate numerical evaluation, if desired. An estimate of its angular dependence in the limit $\Delta \to 2$ can be obtained by replacing the slowly varying $G_3(k_1, k_2)$ by a constant. Then (17) can be evaluated by expanding in terms of spherical harmonics:

$$C_3(\theta_{ij}) \sim \sum_{l_i, m_i} \frac{\mathcal{K}^*_{l_1 m_1 l_2 m_2 l_3 m_3}}{(2l_1+1)(2l_2+1)(2l_3+1)} \left(\frac{1}{l_1+l_2+l_3} + \frac{1}{l_1+l_2+l_3+3} \right) Y_{l_1 m_1}(\hat{r}_1) Y_{l_2 m_2}(\hat{r}_2) Y_{l_3 m_3}(\hat{r}_3) \ , \tag{19}$$

where $\mathcal{K}_{l_1 m_1 l_2 m_2 l_3 m_3} \equiv \int d\Omega Y_{l_1 m_1}(\Omega) Y_{l_2 m_2}(\Omega) Y_{l_3 m_3}(\Omega)$.

In the special case of equal angles $\theta_{ij} = \theta$ [15], it follows from (18) that the three-point correlator is

$$C_3(\theta) \sim (1 - \cos\theta)^{\frac{3}{2}(2-\Delta)} \ . \tag{20}$$

Expanding the function $C_3(\theta)$ in multiple moments as in Eqn. (8) with coefficients $c_\ell^{(3)}$, and normalizing to the quadrupole moment, we find

$$c_\ell^{(3)}(\Delta) = c_2^{(3)}(\Delta) \frac{\Gamma(4 + \frac{3}{2}(2-\Delta))}{\Gamma(2 - \frac{3}{2}(2-\Delta))} \frac{\Gamma(\ell - \frac{3}{2}(2-\Delta))}{\Gamma(\ell + 2 + \frac{3}{2}(2-\Delta))} \ . \tag{21}$$

In the limit $\Delta = 2$, we obtain $\ell(\ell+1)c_\ell^{(3)} = 6c_2^{(3)}$, which is the same result as for the moments $c_\ell^{(2)}$ of the two-point correlator but with a different quadrupole amplitude.

The value of this quadrupole normalization $c_2^{(3)}(\Delta)$ cannot be determined by conformal symmetry considerations alone. A naive comparison with the two-point function which has a small amplitude of the order of 10^{-6} leads to a rough estimate of $c_2^{(3)} \sim \mathcal{O}(10^{-9})$, which would make it very difficult to detect [15]. However, if the conformal invariance hypothesis is correct, then these non-gaussian correlations must exist at some level, in distinction to the simplest inflationary scenarios. Their amplitude is model dependent and possibly much larger than the above naive estimate. The detection of such non-gaussian correlations at any level is therefore an important test for the hypothesis of conformal invariance.

For higher point correlations, conformal invariance does not determine the total angular dependence. Already the four-point function takes the form,

$$\langle \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta(x_2) \mathcal{O}_\Delta(x_3) \mathcal{O}_\Delta(x_4) \rangle \sim \frac{A_4}{\prod_{i<j} r_{ij}^{2\Delta/3}} \ , \tag{22}$$

where the amplitude A_4 is an arbitrary function of the two cross-ratios, $r_{13}^2 r_{24}^2 / r_{12}^2 r_{34}^2$ and $r_{14}^2 r_{23}^2 / r_{12}^2 r_{34}^2$. Analogous expressions hold for higher p-point functions. However in the equilateral case $\theta_{ij} = \theta$, the coefficient amplitudes A_p become constants and the angular dependence is again completely determined. The result is

$$C_p(\theta) \sim (1 - \cos\theta)^{\frac{p}{2}(2-\Delta)} \ , \tag{23}$$

and the expansion in multiple moments yields coefficients $c_\ell^{(p)}$ of the same form as in Eqn. (21) with 3/2 replaced by $p/2$. In the limit $\Delta = 2$, we obtain the universal ℓ-dependence $\ell(\ell+1)c_\ell^{(p)} = 6c_2^{(p)}$.

In summary, the conformal invariance hypothesis applied to the primordial density fluctuations predicts deviations from the Harrison-Zel'dovich spectrum, which should be imprinted on the CMBR anisotropy. A particular realization of this hypothesis is provided by the metric fluctuations induced by the known trace anomaly of massless matter fields which gives rise to fixed point with a spectral index $n > 1$. A second general consequence of conformal invariance is non-gaussian higher point correlations in the statistics of the CMBR. If either of these effects is detected it would be an important clue to the mechanism of the origin of primordial density fluctuations and the formation of structure in the universe.

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